Government of **Western Australia**
Curriculum Council

MATHEMATICS: SPECIALIST

3C/3D
Calculator-assumed

WACE Examination 2011

Marking Key

Marking keys are an explicit statement about what the examiner expects of candidates when they respond to a question. They are essential to fair assessment because their proper construction underpins reliability and validity.

When examiners design an examination, they develop provisional marking keys that can be reviewed at a marking key ratification meeting and modified as necessary in the light of candidate responses.

Section Two: Calculator-assumed **(80 Marks)**

Question 8 (5 marks)

Radium decays at a rate proportional to its present mass; that is, if $Q(t)$ is the mass of radium present at time *t*, then $\frac{dQ}{dt} = kQ$.

It takes 1600 years for any mass of radium to reduce by half.

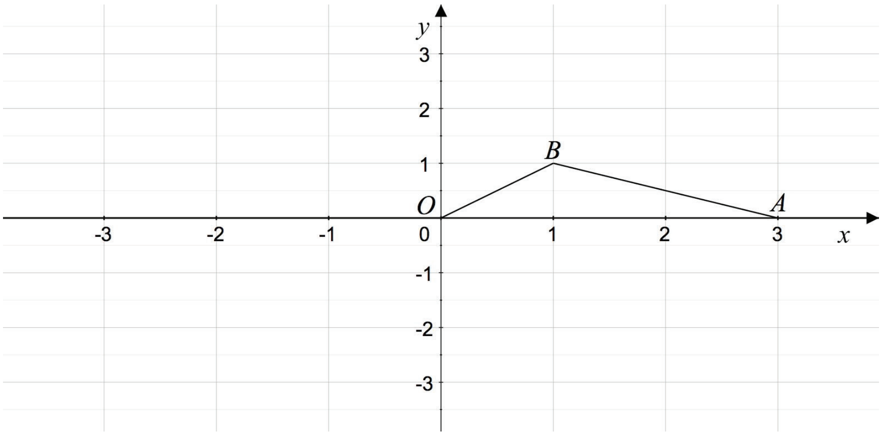
(a) Find the value of k. (3 marks)

Solution
$Q(t) = Ae^{kt}$ $\frac{1}{2} = e^{1600k}$ Hence k = -0.000433 (Accept $\frac{-2\log^2}{1600}$)
Writes the specific behaviours
✓ writes the exponential decay equation ✓ writes an equation for the half-life of radium ✓ solves for k

(b) A factory site is contaminated with radium. The mass of radium on the site is currently five times the safe level. How many years will it be before the mass of radium reaches the safe level? (2 marks)

Solution
Let S be the safe level of radium. Then the initial value satisfies $A = 5S$ i.e. $\frac{1}{5} = e^{\frac{\ln 0.5}{1600}t}$ t = 3715 (Accept 3715 or 3716) It will be 3716 years before the site is safe Or $\frac{1}{5} = e^{-0.000433t}$ t = 3716.95 (Accept 3716 or 3717) It will be 3717 years before the site is safe
Specific behaviours
✓ correctly expresses A in terms of S (or correct ratio) ✓ solves for t

Question 9 **(4 marks)**



A triangle has vertices $O(0,0)$, $A(3,0)$ and $B(1,1)$, as shown in the diagram above.

(a) Write down the matrix that rotates triangle OAB through 90° clockwise about the origin.

(1 mark)

Solution
Required matrix is $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
Specific behaviours
✓ correctly identifies the 2×2 rotation matrix

(b) If triangle OAB is transformed by a dilation about the origin of scale factor k ($k > 0$), determine the matrix which will create an image of area 24 square units. (3 marks)

Solution
Area of triangle OAB = 1.5 square units Area of new triangle is 16 times the area of triangle OAB. i.e. $det\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} = 16$ Hence, required matrix is $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$
Calculates the ratio specific behaviours
✓ calculates the ratio between the areas of shapes before and after dilation ✓ correctly states the dilation matrix in terms of k ✓ solves for k

Or

Solution
Dilation matrix is $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$ Coordinates of dilated triangle are $(0,0)$, $(3k,0)$, (k,k) Hence area of dilated triangle = $\frac{1}{2}\times 3k\times k=\frac{3}{2}k^2$ Hence $\frac{3}{2}k^2=24$ i.e. $k=4$
Specific behaviours
✓ correctly states the dilation matrix in terms of k ✓ uses the new coordinates to determine the area of the dilated triangle in terms of k ✓ solves for k

Question 10 **(8 marks)**

Two radio controlled model planes take off at the same time from two different positions and with constant velocities. Model A leaves from the point with position vector $(-3\boldsymbol{i}-7\boldsymbol{j})$ metres and has velocity $(5\boldsymbol{i}-\boldsymbol{j}+2\boldsymbol{k})$ m/s; model B leaves from the point with position vector $(7\boldsymbol{i}-\boldsymbol{j}-8\boldsymbol{k})$ metres and has velocity $(3\boldsymbol{i}-4\boldsymbol{j}+6\boldsymbol{k})$ m/s.

(a) Find the distance between the two model planes after 1 second of flight. (3 marks)

Solution
(a) $\boldsymbol{r}_A = -3\boldsymbol{i}-7\boldsymbol{j}+t(5\boldsymbol{i}-\boldsymbol{j}+2\boldsymbol{k})$; $\boldsymbol{r}_B = 7\boldsymbol{i}-\boldsymbol{j}-8\boldsymbol{k}+t(3\boldsymbol{i}-4\boldsymbol{j}+6\boldsymbol{k})$ i.e. $\boldsymbol{r}_A = (5t-3)\boldsymbol{i}+(-t-7)\boldsymbol{j}+(2t)\boldsymbol{k}$; $\boldsymbol{r}_B = (3t+7)\boldsymbol{i}+(-4t-1)\boldsymbol{j}+(6t-8)\boldsymbol{k}$ When $t=1$, $\boldsymbol{r}_A = 2\boldsymbol{i}-8\boldsymbol{j}+2\boldsymbol{k}$; $\boldsymbol{r}_B = 10\boldsymbol{i}-5\boldsymbol{j}-2\boldsymbol{k}$ Hence ${}_A\boldsymbol{r}_B = -8\boldsymbol{i}-3\boldsymbol{j}+4\boldsymbol{k}$ Hence distance between the two planes = norm $[-8,\ -3,\ 4]$ = 9.43 metres
Specific behaviours
✓ correctly determines $\boldsymbol{r}_A$ and $\boldsymbol{r}_B$ ✓ determines ${}_A\boldsymbol{r}_B$ when $t=1$ ✓ finds the required distance

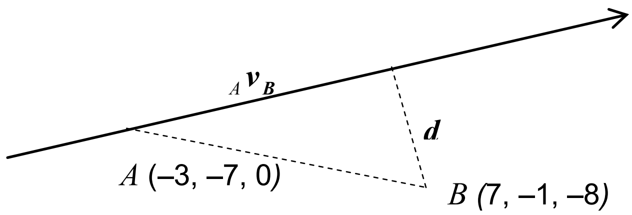
(b) Find: (5 marks)

(i) the shortest distance between the two model planes

(ii) the time when this occurs.

Solution

(i) and (ii)



$$\boldsymbol{d} = \overrightarrow{BA} + \boldsymbol{t}\,{}_{\boldsymbol{A}}\boldsymbol{v_B} = -3\boldsymbol{i} - 7\boldsymbol{j} - (7\boldsymbol{i} - \boldsymbol{j} - 8\boldsymbol{k}) + \boldsymbol{t}(2\boldsymbol{i} + 3\boldsymbol{j} - 4\boldsymbol{k})$$

i.e. $\boldsymbol{d} = (2\boldsymbol{t} - 10)\boldsymbol{i} + (3\boldsymbol{t} - 6)\boldsymbol{j} + (-4\boldsymbol{t} + 8)\boldsymbol{k}$

$\boldsymbol{d} \bullet {}_A\boldsymbol{v_B} = 0$

i.e. $((2\boldsymbol{t} - 10)\boldsymbol{i} = (3\boldsymbol{t} - 6)\boldsymbol{j} + (-4\boldsymbol{t} + 8)\boldsymbol{k}) \bullet (2\boldsymbol{i} + 3\boldsymbol{j} - 4\boldsymbol{k}) = 0$

i.e. $t = \dfrac{70}{29} = 2.41$ seconds

and $|\boldsymbol{d}| = 5.57$ metres

Or $d = (2t - 10)\boldsymbol{i} + (3t - 6)\boldsymbol{j} + (-4t + 8)\boldsymbol{k}$

Hence $|\boldsymbol{d}| = \sqrt{(2\boldsymbol{t} - 10)^2 + (3\boldsymbol{t} - 6)^2 + (-4\boldsymbol{t} + 8)^2}$

Use a calculator to find the minimum value of $|\boldsymbol{d}| = 5.57$ metres

and the value of t for which the minimum occurs:

i.e. $t = \dfrac{70}{29} =$ 2.41 seconds

Specific behaviours

✓ expresses the general distance between the two planes at time t as $\boldsymbol{d} = \overrightarrow{BA} + t\,{}_A\boldsymbol{v_B}$

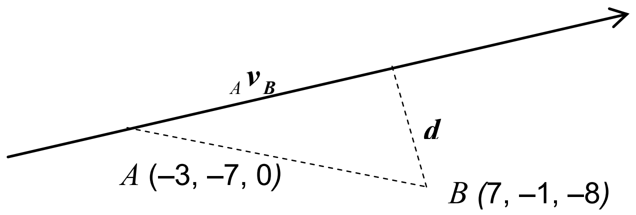
✓ expresses, $\boldsymbol{d}$ in terms of $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ and t

✓ either determines $\boldsymbol{d} \bullet {}_A\boldsymbol{v_B} = 0$ or $|\boldsymbol{d}|$

✓ solves for the minimum value of $|\boldsymbol{d}|$

✓ solves for the corresponding value of t

Or

Solution
(i) and (ii)    Angle between $\overrightarrow{AB}$ and $_A\boldsymbol{v}_B$ from CAS is angle ([10, 6, –8] , [2, 3, –4]) = 23.20°. Hence $\lvert\boldsymbol{d}\rvert = \lvert\overrightarrow{AB}\rvert \times \sin 23.20° = 5.57$ m Also $t \times {}_A\boldsymbol{v}_B = \lvert\overrightarrow{AB}\rvert \times \cos 23.20° = 13.00$ m Hence $t = \dfrac{13.00}{norm[2, 3, -4]} = 2.41$ seconds
Specific behaviours
✓ draws a right triangle with $A, B,\ {}_A\boldsymbol{v}_B$ and $\boldsymbol{d}$ shown ✓ determines the angle between the vectors $\overrightarrow{AB}$ and ${}_A\boldsymbol{v}_B$ ✓ uses the right triangle to determine the length of $\boldsymbol{d}$ ✓ uses the right triangle to determine the length of $t \times {}_A\boldsymbol{v}_B$ ✓ solves for t

Question 11 **(4 marks)**

The triangle ABC has vertices $A(2,1,0)$, $B(3,-3,3)$ and $C(5,0,4)$.

(a) Find the size of $\angle ABC$ correct to the nearest degree. (2 marks)

Solution
$\overrightarrow{BA} = -\boldsymbol{i} + 4\boldsymbol{j} - 3\boldsymbol{k}$; $\overrightarrow{BC} = 2\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}$ Hence angle ([–1, 4, –3] , [2 ,3, 1]) = $\angle ABC = 68^\circ$ (using a CAS)
Specific behaviours
✓ determines the components of vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$ ✓ calculates the required angle

(b) Given that the vector $\left(-13\boldsymbol{i} + 5\boldsymbol{j} + 11\boldsymbol{k}\right)$ is perpendicular to the plane which contains the triangle ABC, find the vector equation of this plane. (2 marks)

Solution
Vector equation of the plane is $\left(\boldsymbol{r} - (2\boldsymbol{i} + \boldsymbol{j})\right) \bullet \left(-13\boldsymbol{i} + 5\boldsymbol{j} + 11\boldsymbol{k}\right) = 0$ Or $\left(\boldsymbol{r} - (3\boldsymbol{i} - 3\boldsymbol{j} + 3\boldsymbol{k})\right) \bullet \left(-13\boldsymbol{i} + 5\boldsymbol{j} + 11\boldsymbol{k}\right) = 0$ Or $\left(\boldsymbol{r} - (5\boldsymbol{i} + 4\boldsymbol{k})\right) \bullet \left(-13\boldsymbol{i} + 5\boldsymbol{j} + 11\boldsymbol{k}\right) = 0$ Or $\boldsymbol{r} \bullet \left(-13\boldsymbol{i} + 5\boldsymbol{j} + 11\boldsymbol{k}\right) = -21$
Specific behaviours
✓ determines a general vector in the plane ✓ correctly determines the plane equation

Question 12 **(6 marks)**

Three dry cleaning outlets, A, B and C compete for business. Each year A loses 40% of its customers to B and 20% to C; B loses 30% to A, 50% to C; C loses 60% to A, 10% to B.

(a) Complete the following transition matrix. (2 marks)

$$\begin{array}{cc|ccc|} & & & \text{From} & \\ & & A & B & C \\ \hline & A & 0.4 & \_\_\_\_ & \_\_\_\_ \\ \text{To} & B & 0.4 & \_\_\_\_ & \_\_\_\_ \\ & C & 0.2 & \_\_\_\_ & \_\_\_\_ \\ \hline \end{array}$$

Solution

$$\begin{bmatrix} 0.4 & 0.3 & 0.6 \\ 0.4 & 0.2 & 0.1 \\ 0.2 & 0.5 & 0.3 \end{bmatrix}$$

Specific behaviours

✓✓ correctly completes the transition matrix

Or

✓ partially correct (at least four(4) correct)

(b) At the end of 2011, company A will have 80% of market share, while B and C will have 10% each. What will be the market share of each company at the end of 2012? (2 marks)

Solution

$$\begin{bmatrix} 0.4 & 0.3 & 0.6 \\ 0.4 & 0.2 & 0.1 \\ 0.2 & 0.5 & 0.3 \end{bmatrix} \times \begin{bmatrix} 80 \\ 10 \\ 10 \end{bmatrix} = \begin{bmatrix} 41 \\ 35 \\ 24 \end{bmatrix}$$

Hence A will have 41%, B will have 35% and C will have 24%.

Specific behaviours

✓ sets up column matrix for market share

✓ accurately multiplies transition matrix with market share matrix

(c) If these conditions remain unchanged, what will be the long-term percentage market share for each company, correct to **one (1)** decimal place? (2 marks)

Solution

$$\begin{bmatrix} 0.4 & 0.3 & 0.6 \\ 0.4 & 0.2 & 0.1 \\ 0.2 & 0.5 & 0.3 \end{bmatrix}^{20} \times \begin{bmatrix} 80 \\ 10 \\ 10 \end{bmatrix} = \begin{bmatrix} 43.6 \\ 25.6 \\ 30.8 \end{bmatrix}$$

Hence A will have 43.6%, B will have 25.6% and C will have 30.8%.

Specific behaviours

✓ chooses a suitably large index for the transition matrix to ensure stability

✓ states the value of each company’s share

Question 13 **(6 marks)**

An engine piston undergoes simple harmonic motion which can be described by the differential equation $\frac{d^2x}{dt^2} = -9x$, where x m is the displacement of the piston from its mean position at t seconds.

(a) Write down the period of the motion. (1 mark)

Solution
$n^2 = 9$ where n is the angular velocity Hence the period of motion is $\frac{2\pi}{3}$ seconds
Specific behaviours
✓ correctly defines the period

(b) If the maximum speed of the piston is 5 m/s, find the amplitude of the motion. (2 marks)

Solution
$v_{\max} = An$ where A is the amplitude Hence $A = \frac{5}{3}$ metres
Specific behaviours
✓ uses the equation $v_{\max} = An$ or $v^2 = n^2(A^2 - x^2)$ at $x = 0$ ✓ correctly solves for A

(c) The amplitude and period of the motion are now changed, but the piston still undergoes simple harmonic motion. These new readings are taken:

when $x = 1$ m, speed $= \sqrt{60}$ m/s; when $x = 3$ m, speed $= \sqrt{28}$ m/s

Find the new exact values for: (3 marks)

(i) the period.

(ii) the amplitude.

Solution
$v^2 = n^2\left(A^2 - x^2\right)$ Hence: $60 = n^2\left(A^2 - 1\right)$ and $28 = n^2\left(A^2 - 9\right)$ Solving gives n = 2 and A = 4 Hence: (i) period = π seconds (ii) amplitude = 4 metres
Specific behaviours
✓ correctly uses the equation $v^2 = n^2\left(A^2 - x^2\right)$ ✓ uses a CAS to solve for n ✓ uses a CAS to solve for A

Question 14 **(5 marks)**

The points P, Q and R are such that $\overrightarrow{PQ} = 5\boldsymbol{i}$ and $\overrightarrow{PR} = \boldsymbol{i} + 4\boldsymbol{j} + 2\boldsymbol{k}$.
Find the vector $\overrightarrow{RM}$ which is parallel to $\overrightarrow{PQ}$ and such that the size of $\angle RQM$ is 90º.

Solution
Let $\overrightarrow{RM} = \lambda\boldsymbol{i}$ for some real number λ $\overrightarrow{QM} = \overrightarrow{QP} + \overrightarrow{PR} + \overrightarrow{RM} = -5\boldsymbol{i} + \boldsymbol{i} + 4\boldsymbol{j} + 2\boldsymbol{k} + \lambda\boldsymbol{i} = (-4+\lambda)\boldsymbol{i} + 4\boldsymbol{j} + 2\boldsymbol{k}$ If angle RQM is 90º, then $\overrightarrow{QM} \bullet \overrightarrow{QR} = 0$ i.e. $((-4+\lambda)\ \boldsymbol{i} + 4\boldsymbol{j} + 2\boldsymbol{k}) \bullet (-4\boldsymbol{i} + 4\boldsymbol{j} + 2\boldsymbol{k}) = 0$ i.e. $\lambda = 9$ so $\overrightarrow{RM} = 9\boldsymbol{i}$
Specific behaviours
✓ uses parallelism to define $\overrightarrow{RM}$ ✓ expresses $\overrightarrow{QM}$ in terms of $\overrightarrow{PQ}$, $\overrightarrow{PR}$, and $\overrightarrow{RM}$ ✓ simplifies in terms of $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$ ✓ equates the dot product of perpendicular vectors to zero ✓ solves for λ and hence $\overrightarrow{RM}$

Question 15 **(5 marks)**

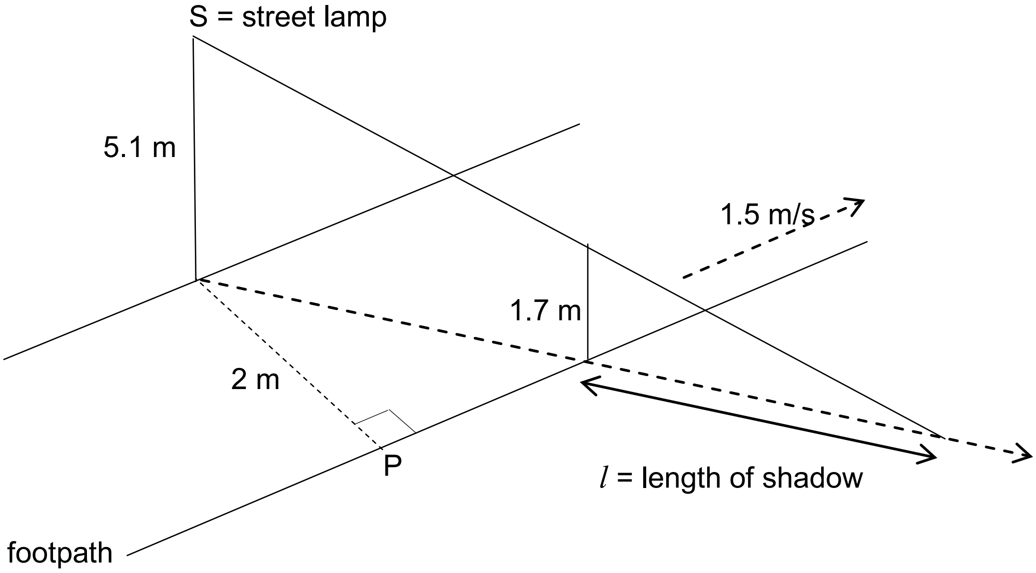
(a) Use Euler's formula ($e^{ix} = \cos x + i\sin x$) to show that $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$. (3 marks)

Solution
$\dfrac{e^{ix} - e^{-ix}}{2i} = \dfrac{(\cos x + i\sin x) - (\cos x - i\sin x)}{2i}$ i.e. $\dfrac{e^{ix} - e^{-ix}}{2i} = \dfrac{2i\sin x}{2i} = \sin x$
Specific behaviours
✓ rewrites e^{ix} as $\cos x + i\sin x$ ✓ rewrites e^{-ix} as $\cos x - i\sin x$ ✓ correctly simplifies

(b) Expand $\left(\dfrac{e^{ix} - e^{-ix}}{2i}\right)^5$ to obtain an expression for $\sin^5 x$ in terms of $\sin x$, $\sin 3x$ and $\sin 5x$. (2 marks)

Solution
cexpand$\left(\dfrac{e^{ix} - e^{-ix}}{2i}\right)^5 = \dfrac{5\sin x}{8} - \dfrac{5\sin 3x}{16} + \dfrac{\sin 5x}{16}$ i.e. $\sin^5 x = \dfrac{5\sin x}{8} - \dfrac{5\sin 3x}{16} + \dfrac{\sin 5x}{16}$ Note: There will be students who initially expand the bracket to get: $\dfrac{1}{32i}\left(e^{5ix} - 5e^{3ix} + 10e^{ix} - 10e^{-ix} + 5e^{-3ix} - e^{-5ix}\right)$ and continue the expansion with $\sin x$ which is acceptable if correct.
Specific behaviours
✓ rewrites $\sin^5 x$ as $\left(\dfrac{e^{ix} - e^{-ix}}{2i}\right)^5$ ✓ uses a CAS calculator to expand $\left(\dfrac{e^{ix} - e^{-ix}}{2i}\right)^5$ to give the required result

Question 16 **(6 marks)**



In the diagram above, P is the initial position of a boy, of height 1.7 metres, who is walking along a straight footpath in the direction shown.

S is the position of a street lamp of height of 5.1 metres; its base is 2 metres from P.

The street lamp will cast a moving shadow of the boy as he continues to walk along the footpath at 1.5 m/s.

(a) If x metres is the distance walked by the boy, show that the length (l metres) of the boy's shadow is $l = \frac{1}{2}\sqrt{4+x^2}$. (3 marks)

Solution
The hypotenuse of the triangle right-angled at P is $\sqrt{4+x^2}$ Then $\frac{1.7}{l} = \frac{5.1}{l+\sqrt{4+x^2}}$ (using similar triangles) i.e. $l = \frac{1}{2}\sqrt{4+x^2}$
Specific behaviours
✓ expresses the hypotenuse of the right triangle in terms of x ✓ uses similar triangles to determine an equation in x and l ✓ simplifies correctly to express l in terms of x

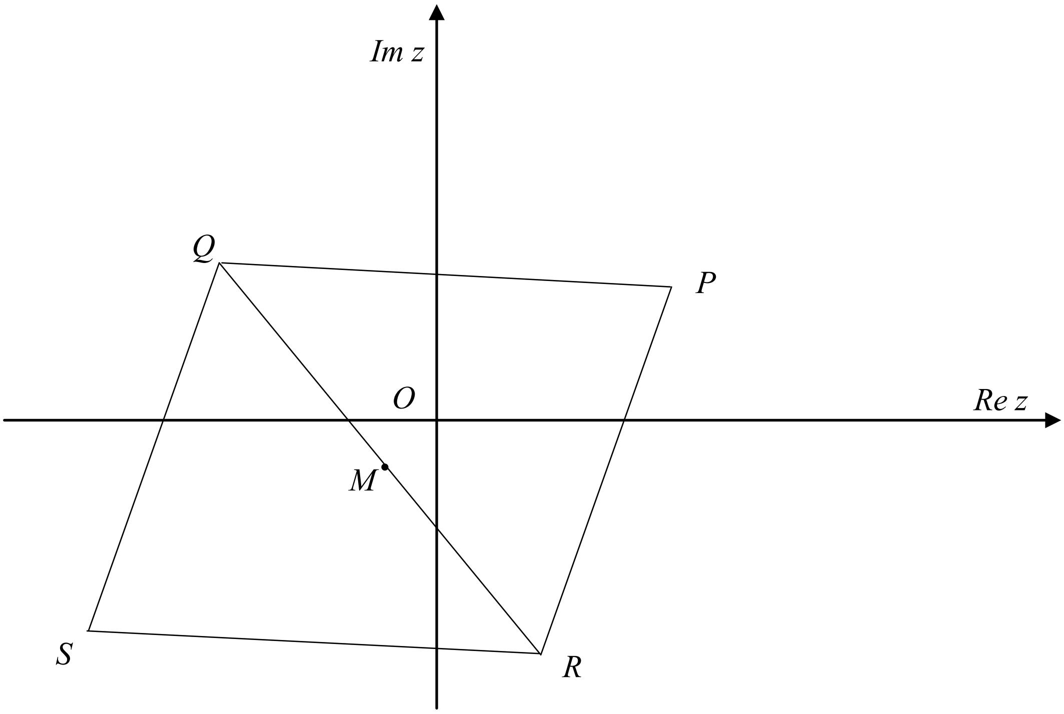
(b) Find the rate of change, in m/s, of the length of the boy's shadow after 5 seconds.

(3 marks)

Solution
$l = \frac{1}{2}\sqrt{4+x^2}$ Hence $\frac{dl}{dt} = \frac{dl}{dx} \times \frac{dx}{dt} = \frac{x}{2\sqrt{4+x^2}} \times \frac{dx}{dt}$ i.e. $\frac{dl}{dt} = \frac{3x}{4\sqrt{4+x^2}}$ since $\frac{dx}{dt} = 1.5$ When t = 5, x = 7.5 Hence $\frac{dl}{dt} = \frac{3 \times 7.5}{4\sqrt{4+7.5^2}} = 0.72$ m/s
Specific behaviours
✓ differentiates l with respect to x ✓ uses chain rule with $\frac{dx}{dt} = 1.5$ ✓ carries through calculation accurately

Question 17 **(9 marks)**

The point P on the Argand diagram below represents the complex number z. The points Q and R represent the points wz and $\overline{w}z$ respectively, where $w = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$. The point M is the midpoint of QR. (The diagram is not drawn to scale.)



(a) If $z = rcis(\theta)$, find wz and $\overline{w}z$ in polar form. (2 marks)

Solution
$wz = rcis\left(\theta + \frac{2\pi}{3}\right)$ $\overline{w}z = rcis\left(\theta - \frac{2\pi}{3}\right)$
Specific behaviours
✓ writes wz in polar form ✓ writes $\overline{w}z$ in polar form

(b) Hence explain why $\left|\overrightarrow{OP}\right| = \left|\overrightarrow{OQ}\right| = \left|\overrightarrow{OR}\right|$. (2 marks)

Solution
$\overrightarrow{OP} = rcis(\theta)$ $\overrightarrow{OQ} = rcis\left(\theta + \frac{2\pi}{3}\right)$ $\overrightarrow{QR} = rcis\left(\theta - \frac{2\pi}{3}\right)$ $\therefore \left|\overrightarrow{OP}\right| = \left|\overrightarrow{OQ}\right| = \left|\overrightarrow{OR}\right| = r$
Specific behaviours
✓ expresses the vectors as complex numbers ✓ states $\operatorname{mod} z = \operatorname{mod} wz = \operatorname{mod} \overline{w}z = r$

(c) Show that the complex number representing M is $-\frac{1}{2}z$. (2 marks)

Solution
$OM = \frac{1}{2}\overrightarrow{OQ} + \frac{1}{2}\overrightarrow{OR}$ $= \frac{1}{2}\left(wz + \overline{w}z\right)$ $= \frac{1}{2}z\left(cis\frac{2\pi}{3} + cis\left(\frac{-2\pi}{3}\right)\right)$ $= \frac{1}{2}z\left(2\cos\frac{2\pi}{3}\right)$ $= -\frac{1}{2}z$
Specific behaviours
✓ correctly defines $\overrightarrow{OM}$ as vector terms ✓ simplifies the expression using polar form

(d) The point S is chosen so that $PQSR$ is a parallelogram. Find the complex number represented by S in terms of z. (3 marks)

Solution

$$\begin{aligned}\overrightarrow{OS} &= \overrightarrow{OM} + \overrightarrow{MS} \\ &= \overrightarrow{OM} - \overrightarrow{MP} \\ &= \overrightarrow{OM} - \left(\overrightarrow{MO} + \overrightarrow{OP}\right) \\ &= 2\overrightarrow{OM} - \overrightarrow{OP} \\ &= 2\left(-\frac{1}{2}z\right) - z \\ &= -2z\end{aligned}$$

Specific behaviours

✓ correctly defines $\overrightarrow{OS}$ in vector terms

✓ converts from vector terms to complex numbers

✓ correctly simplifies

Question 18 **(8 marks)**

A model for a population, P, of numbats is

$P = \dfrac{900}{3+2e^{-t/4}}$, where *t* is the time in years from today.

(a) What is the population today? (1 mark)

Solution
Population today = $\dfrac{900}{3+2} = 180$
Specific behaviours
✓ sets $t = 0$ and solves for P

(b) What does the model predict that the eventual population will be? (1 mark)

Solution
Eventual population = $\dfrac{900}{3} = 300$
Specific behaviours
✓ lets $t \to \infty$ and solves for P

(c) By first expressing $e^{-t/4}$ in terms of P, or otherwise, show that P satisfies the differential equation $\frac{dP}{dt} = \frac{P}{4}\left(1 - \frac{P}{300}\right)$. (4 marks)

Solution
$e^{-t/4} = \frac{450}{P} - \frac{3}{2}$ Hence $-\frac{e^{-t/4}}{4} = -\frac{450}{P^2} \times \frac{dP}{dt}$ i.e. $\frac{1}{4} \times \left(\frac{450}{P} - \frac{3}{2}\right) = \frac{450}{P^2} \times \frac{dP}{dt}$ Hence $\frac{dP}{dt} = \frac{P^2}{4 \times 450} \times \left(\frac{450}{P} - \frac{3}{2}\right)$ i.e. $\frac{dP}{dt} = \frac{P}{4}\left(1 - \frac{P}{300}\right)$
Specific behaviours
✓ correctly rearranges the equation ✓ differentiates $\left(e^{-t/4} = \frac{450}{P} - \frac{3}{2}\right)$ implicitly with respect to t ✓ substitutes $\left(e^{-t/4} = \frac{450}{P} - \frac{3}{2}\right)$ to give an equation for $\frac{dP}{dt}$ involving P only ✓ rearranges and simplifies

(d) What is the instantaneous percentage annual rate of growth today? (2 marks)

Solution
The instantaneous rate of change today = $\frac{dP}{dt}$ when $t = 0$ and $P = 180$ i.e. $\left.\frac{dP}{dt}\right\|_{t=0} = 18$ Hence the instantaneous percentage rate of growth today = $\frac{18}{180} \times 100 = 10\%$
Specific behaviours
✓ calculates the instantaneous rate of change today ✓ calculates the required percentage

Question 19 **(7 marks)**

Let $f\left(n\right)=3^{n+2}+\left(-1\right)^{n}\times 2^{n}$, for all positive integers n.

(a) Show that $2f\left(n+1\right)-f\left(n\right)$ is divisible by 5. (2 marks)

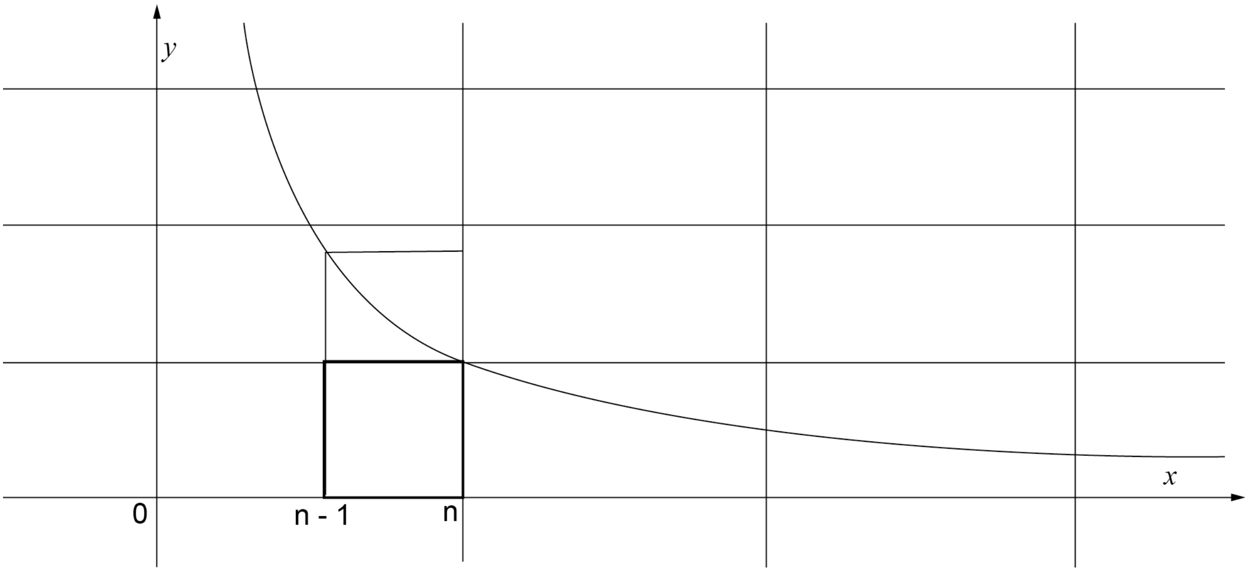
Solution
$2f\left(n+1\right)-f\left(n\right)=2\left(3^{n+3}+\left(-1\right)^{n+1}\times 2^{n+1}\right)-3^{n+2}-\left(-1\right)^{n}\times 2^{n}$ Using a CAS the RHS simplifies to $45\times 3^{n}-5\times\left(-2\right)^{n}$ Hence result
Specific behaviours
✓ correctly expands $2f\left(n+1\right)-f\left(n\right)$ ✓ simplifies to correct term with factor of 5

(b) Hence, or otherwise, prove by induction that $f(n)$ is divisible by 5. (5 marks)

Solution
Let $P\left(n\right)$ be the statement $f\left(n\right)=3^{n+2}+\left(-1\right)^{n}\times 2^{n}=5s$ for some integer s. $P\left(1\right)$ is true because $3^{3}-2=25=5\times 5$. Assume $P\left(k+1\right)$ is true. i.e. Assume $f\left(k\right)=5w$ for some integer w. Consider $P\left(k+1\right)$. Required to show that $f\left(k+1\right)=5p$ for some integer p. From part (i), $2f\left(k+1\right)-f\left(k\right)=5t$ for some integer t. Hence $2f\left(k+1\right)=f\left(k\right)+5t=5w+5t=5\left(w+t\right)$ using the induction assumption Hence $f\left(k+1\right)=5p$ for some integer p, since 2 is not divisible by 5 Thus if $P\left(k\right)$ is true, then $P\left(k+1\right)$ is also true. But $P\left(1\right)$ is true. Hence $P\left(n\right)$ is true for all $n\geq 1$
Specific behaviours
✓ shows that $P\left(1\right)$ is true ✓ states the induction assumption ✓ shows that $2f\left(k+1\right)$ is divisible by 5 if $P\left(k\right)$ is true ✓ justifies that this proves that $f\left(k+1\right)$ is divisible by 5 if $P\left(k\right)$ is true ✓ makes a final statement which explains why this is a valid proof by induction

Question 20 **(7 marks)**

Let n be a positive integer greater than 1. The area of the region under the curve $y = \frac{1}{x}$ from $x = n - 1$ to $x = n$ lies between the areas of the two rectangles, as shown in the diagram.



(a) Use the diagram to show that $e^{-n/(n-1)} < \left(1-\frac{1}{n}\right)^n < e^{-1}$. (6 marks)

Solution

Area of the larger rectangle is $\frac{1}{n-1}$ sq units; area of the smaller rectangle is $\frac{1}{n}$ sq units

Hence $\frac{1}{n} < \int_{n-1}^{n} \frac{1}{x}\,dx < \frac{1}{n-1}$

i.e. $\frac{1}{n} < [\ln x]_{n-1}^{n} < \frac{1}{n-1}$

i.e. $\frac{1}{n} < \ln\left(\frac{n}{n-1}\right) < \frac{1}{n-1}$

Hence $e^{\frac{1}{n}} < \frac{n}{n-1} < e^{\frac{1}{n-1}}$

Hence $\frac{1}{e^{\frac{1}{n-1}}} < \frac{n-1}{n} < \frac{1}{e^{\frac{1}{n}}}$

Hence $\frac{1}{e^{\frac{n}{n-1}}} < \left(\frac{n-1}{n}\right)^n < \frac{1}{e^{\frac{n}{n}}}$

Hence $e^{-\frac{n}{n-1}} < \left(1-\frac{1}{n}\right)^n < e^{-1}$

Specific behaviours

- ✓ identifies that $\int_{n-1}^{n} \frac{1}{x}\,dx$ lies between the areas of the two rectangles
- ✓ integrates and simplifies to establish $\frac{1}{n} < \ln\left(\frac{n}{n-1}\right) < \frac{1}{n-1}$
- ✓ uses the inverse relationship between $\ln x$ and e^x
- ✓ inverts the fractions
- ✓ recognises the need to reverse the order of the inequalities
- ✓ raises each term to the power n

(b) Hence deduce $\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n$. (1 mark)

Solution

$\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n = e^{-1}$

Specific behaviours

- ✓ uses the pinching theorem to establish the limit